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THE EFFORT TO MAKE ALGEBRA YIELD FRUIT.

By W. A. Cornish.

(*Continued from p. 159.*)

If this analysis is correct it has important bearings on the world-wide effort to give concreteness to algebra. Concreteness as Smith and McMurry have pointed out is to be secured by solving problems where they are found, in their home environment. Abstractness is the sure result of taking things out of their home environment and treating them by themselves.

The home surroundings of algebra are geometry, physics, and arithmetic. Algebra has meaning and use only as it keeps intact its connection with these other branches just as mathematics as a whole has meaning and value only as it keeps intact its connection with nature.

The inference seems to be that these subjects should not be isolated and studied, first arithmetic for a while, then algebra for another while, then geometry for a third while, then reviewed and gone on with one at a time, in another rotation; but that the work ought to be carried on simultaneously in all these subjects as is done on the continent and in Great Britain.

We thought that this was what our brethren of the Central Association were striving for, but after all their violent thrashing about for the last ten years their committee reports that the work in high school algebra should consist of an elementary

course given the first year, and a more advanced course given the third year after demonstrational geometry, which is to come presumably during the second year.

If the people of the Central Association, who have been leading in the fight for reform, have made up their minds to consent for another lapse of time to the American plan of isolating the branches of mathematics in the high school course, there is probably no use in trying for a rearrangement here. With us of the conservative east, the thing that has been is, right or wrong, the thing that shall be.

And perhaps it is just as well, for in demonstrative geometry, since the time of Euclid there has been a sacrifice of subject-matter to logical method. The question has not been, what are the principles of geometry and what are they good for, but why are they true and what are the logical processes by which they are established. Geometry has been primarily an exercise book in logic and only incidentally a branch of mathematics. The traditional sequence of propositions that has resulted makes geometry rather ill-suited to be a companion book for algebra and source of algebraic material. In the first and second books there are a few definitions and principles leading to interesting algebraic problems, but they are not adequate for the purpose of illustrating and concreting algebra.

Algebra needs from the start not only lines and angles, but surfaces and solids, otherwise the terms linear, quadratic, cubic, have no meaning. The geometric principles out of which algebra most naturally grows are those pertaining to areas of plane figures and volumes of solids, similar figures both plane and solid, the relations of sides of triangles and of chords, secants, and tangents of circles. The increasing study of concrete geometry in the upper grades continued in connection with algebra in the first year of the high school may furnish a really better solution of the problem of giving concreteness to algebra than the simultaneous study of algebra and demonstrative geometry.

The conclusion then is as follows:

The suggestions of the committee of the Central Association on algebra in the secondary schools are excellent, entirely feasible, and justifiable on philosophic grounds, and ought to be endorsed and adopted in toto. And farther than that the work

in algebra during the first year of the high school should be accompanied by a concrete development of those principles of plane and solid geometry that furnish the basis of practical algebra.

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INTUITION AND LOGIC IN GEOMETRY.*

BY W. BETZ.

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Geometry has been called a "mixture of logic and intuition, with logic predominating" (M. Simon).

This naturally suggests an investigation concerning the relative importance of these two factors in the development of the subject.

First, of all, then, I must define my terms. What is logic? What is intuition? I will define the first as the science of reasoning (or of *deduced* inference), the second as the principle of direct insight (or *immediate* inference).

Now, there are many kinds of intuition. It may be said that to each of our instincts or talents there corresponds a form of intuition. Thus we may speak of musical intuition, mechanical intuition, etc. I am here concerned with that form of intuition which enters into our mathematical work. As such, it seems to have two aspects. It is either *arithmetic* (corresponding to our number instinct) or *geometric* (corresponding to our instinct of form).

Any form of intuition may be abnormally developed. A great composer has an unusual musical intuition. It has been pointed out that very few individuals possess a highly developed arithmetic intuition. Geometric intuition is far more prevalent. This raises the question: Are these forms of intuition of the

same value, from a *scientific* standpoint? Moreover, does mathematics really require two forms of intuition?

Let us consider some illustrations of arithmetic and geometric intuition.

It seems instinctively clear to a pupil that $2+2=4$, that a units plus b units is the same as b units plus a units, that if two points are on opposite sides of a line, the segment joining them crosses the line, etc. What the *origin* of these concepts is, does not concern us now. But it will be observed directly that the arithmetic symbols simply *suggest* the idea; they do not contain the idea. The fact that $2+2=4$ is stated without appeal to the senses. Arithmetic may thus be regarded as a purely mental, logical discipline. In geometry, however, whenever a diagram, physical or mental, is used, the geometric ideas often come from a visualization of the figure.

It may well be questioned, of course, whether the fundamental ideas of number do not themselves arise through sense-perception. Even if this be granted there is still this important difference between ordinary arithmetic and ordinary geometry that the symbolism of geometry constantly appeals to the senses, while that of arithmetic does not.

If, then, the relative scientific merit of arithmetic and geometry is to be determined, we must really ask: Since ordinary geometry appeals to the senses, can the senses be trusted *always* to give *certainty?* Are there recorded cases of erroneous conclusions owing to this guidance of the senses? If so, what is the attitude of mathematical science toward geometric intuition? Is this attitude binding on the secondary schools?

Following this line of thought, my subject naturally divides itself into three parts. I wish to show:

1. That pure mathematics tends to *exclude* space intuition, and why,

2. That the school must retain and even cultivate space intuition, and why,

3. Precisely which elements space intuition has contributed and is contributing to the geometry of the school.

I. My first topic is historical. Why does pure mathematics oppose space intuition and how can space intuition be eliminated?

To answer this question, a long introduction is necessary.

The existence of geometry has always been more or less of a puzzle to thinking men. Here is a science that apparently pertains to *external* things, but whose results seem to be independent of these things. We reason concerning these external things, and the science progresses by adding inference to inference, apparently by abstract reasoning; and—strange to say—experience seems to agree with the results of our reasoning, even though we seem to pay not the slightest attention to the external world.

This is certainly very curious, and it has led to various attempts at explanation. The latter may be divided into two classes: the empirical theories, and the *a priori* theory.*

The empiricist says that our spatial notions are but reminiscences of forms we have seen or touched. We can join these mental images together and even change them (idealize them), but their origin is unquestionably in some physiological sensation. Hence, even mental intuition comes from experience. This may be said to be the view of many prominent scientists of the present day. It has been set forth very elaborately by *Mach* in his "Space and Geometry." It was the view of Newton and Leibnitz (cf. Hölder). According to this idea, the fundamental notions and axioms of geometry are the direct consequences of innumerable physical observations, comparisons and measurements. In the language of Mach: "Geometry is concerned with *ideal* objects produced by the schematization of *experimental* objects." It was this conception, undoubtedly, that led Helmholtz to speak of physical geometry.

Directly opposed to this empirical view is the *a priori* theory, which found its classical representative in Kant. A rigid follower of Kant would object to the idea of physical measurement on the ground that before we begin to measure, ideas of length, of equality, preexist in the mind, and that we are arguing in a circle in ascribing to these notions a physical origin. Kant asserted that "one could never picture to oneself that space did not actually exist, although one might quite easily imagine that there were no objects in space." Kant's space arguments are no longer accepted by philosophy.†

It is easy to understand how Kant arrived at his theory.

* Cf. Hölder, "Anschauung und Denken in der Geometrie."

† Cf. O. Külpe, "Immanuel Kant," Leipzig, 1907, pp. 48–57.

The skepticism of Hume had led him to doubt the messages of his senses. They could give only subjective, not universal, certainty, and hence one could never know the "thing in itself" (das Ding an sich). Accordingly, if geometry is based on sense-perception, it rests on a very doubtful foundation and cannot compel universal acceptance. This conclusion was unbearable to Kant. If he felt sure of anything, it was of the certainty of mathematical knowledge. In particular, he pointed out Euclid's geometry as a model of certainty. It had endured the examination of twenty centuries and seemed as valid as ever. If, then, geometry is certain, its data cannot come from the senses, but must come from the mind. And so he built up his famous *a priori* theory. Space is a necessary *form* of perception. Prior to perception there exists in the mind a consciousness of space without which experience would be impossible. Mathematical knowledge, like every other knowledge, begins with experience, but does not depend on experience. Thoughts without content are empty, perceptions without conceptions are blind. Perception and conception must go together. Understanding alone can perceive nothing, the senses alone can think nothing. Knowledge arises from their unified action. Yet they must not be confused. It is necessary to separate perception from *pure intuition,* and both of these from understanding or logic.

As stated before, the investigations of modern psychologists and physiologists have shown that our notions of space are presumably due to experience. They are caused primarily by visual and tactual sensations.* Professor William James even advanced the theory that *every* sensation is in part spatial in character. In fact, the physiological element in geometry can hardly be ignored. We are reasonably certain that our conception of space is very composite, being due to many stimuli, but that we may overlook or ignore the particular stimuli producing a given spatial motion.†

* Cf. Mach's "Space and Geometry," pp. 84, 85, 86, 88, 89, 90, 92, 93, 142, etc. Helen Keller's space sensations are described by her in recent numbers of *Century*.

† Hence, "if our sensations of space are independent of the quality of the stimuli which go to produce them, then we may make predications concerning the former independently of external or physical experience. It is the imperishable merit of Kant to have called attention

These investigations seem to explode Kant's *a priori* theory, and give the *first* reason why modern pure mathematics rejects space intuition.

The *second* reason is offered by the history of geometry itself.

No other subject has such an instructive and elaborate history. It may suffice here to say that after many centuries of preliminary development, in Babylonia and Egypt, geometry became a science at the hands of Greek scholars, assuming its crowning form in the "Elements of Euclid." Euclid's vast work was regarded by his contemporaries as perfect, in spite of its many appeals to intuition. It seemed to meet their requirements of rigor.

To-day we still admire Euclid's text, but we do not regard it as a model of rigor. In particular, we object to his definitions and the formulation of his axioms, *i. e.,* to his geometric foundation. It has taken 2,000 years to produce a better insight into the elements of geometry. Only the principal facts in this evolution can be mentioned here.

Euclid placed at the beginning of his system certain definitions, postulates and "common notions." His postulates were five in number:

1. A straight line can be drawn connecting any two points.
2. A segment may be continued indefinitely.
3. A circle may be drawn with any radius and any center.
4. All right angles are equal.
5. If, when two straight lines are crossed by a transversal, the sum of the interior angles on the same side of the transversal is less than two right angles, the two lines will meet on the side where the sum of the angles is less than two right angles.

Postulate 5 has become famous as the *parallel postulate.* In a certain sense, its history is that of geometry since the time of Euclid. Excellent accounts of this history have been given. I refer especially to the work of Engel and Stäckel.

Mr. Withers has written a monograph on the parallel postulate. A good summary account is given by Mach.* The history of this postulate may, according to Withers, be divided

to this point." (Mach, p. 34). "But this basis is *unquestionably* inadequate to the complete development of a geometry, inasmuch as concepts, and in addition thereto concepts derived from experience, are *also* requisite to this purpose."

* "Space and Geometry," pp. 114–142.

into two periods characterized, respectively, (1) by attempts to dispense with the postulate either by substituting a different definition of parallel lines or by substituting a different formulation of the postulate, (2) by direct attempts at demonstration. The names of Ptolemy, and later Saccheri, Lambert, Gauss, Legendre and others, belong to this movement.

All these efforts resulted in failure. Finally, it dawned on a few chosen minds to study the effect of omitting this postulate or of replacing it by its contradiction. If either procedure, so they agreed, still led to a logically consistent result, then the parallel postulate could not be a *necessary* assumption. The experiment was entirely successful and produced a revolution in geometry. For it robbed Euclid's geometry of its exclusive character and destroyed another of Kant's arguments. The nineteenth century brought the discovery and development of these "non-Euclidean" geometries. The names most prominently connected with this development are those of Lobatchefski, Bolyai, Gauss, Riemann, Helmholtz, Clifford, Klein, Lie and others.

It is part of the irony of history that while Kant was pondering over his *a priori* theory, J. Heinrich Lambert—whom Kant called "the incomparable man"—was helping to demolish Kant's chief argument. "As late as 1799 Gauss was still trying to prove *a priori* the reality of Euclid's system. But in 1829 he wrote to Bessel, stating that his conviction that we cannot found geometry completely *a priori* had become, if possible, still firmer, and that if number is completely the product of our mind,* space has a *reality beyond our mind* of which we cannot foreordain the laws *a priori*."

Gauss saw that a new era had begun. Things moved rapidly. And with the appearance of other logically consistent geometric systems a different conception of the foundations of the subject became necessary.

It is interesting to ask why the *parallel* postulate, rather than some other axiom, should have been the chief disturbing element in Euclid's geometry. The answer is that of all the postulates it alone did not seem clear *intuitively*. This "defect" was soon felt by the followers of Euclid. It was this feeling that led Saccheri to write his now famous "Euclides ab omni *nævo* vin-

* Cf. Mach, p. 98.

dicatus" (Milan, 1733). All these attempts to rid Euclid's system of its only "blemish," clearly show that the eyes of the mathematicians had not been opened to see the essentially intuitive character of the entire structure they were trying to perfect. The parallel postulate finally rendered this service to the science.

Now, are all these new geometries equally true? Logically, yes. Are they all of the same practical value? No. The philosophic aspect of the problem is, in fact, still unsettled. With which of these geometries our space agrees, we cannot say definitely. We may never know this. It is true that the most refined measurements all point to Euclid's system as the geometry of our experience. If, however, it should become possible—some day—to prove our space absolutely Euclidean, it would not destroy the logically valid character of the non-Euclidean geometries. For we should regard them, even then, as possible and justifiable speculations proving our spatial world not to be the only conceivable one. (Cf. Weber and Wellstein).

We are compelled to say, then, that Euclid's system must abandon the claim to the exclusive validity of its foundations, and can only assert the correctness of the inferences based on these.

All this does not mean that the place of geometry in the school is endangered. The space notion is so important that if our universe were suddenly transformed into a non-Euclidean space, we should simply study the geometry of that space.

However, the effect on the critical examination of the foundations has been enormous. The breakdown of intuition as a tool of investigation—as illustrated by the history of the parallel postulate—has taught the modern pure mathematicians to reduce its use to a minimum. For, intuition cannot give *certainty*. Hence, if geometry is to retain the character of a science, it must change its form.

This transformation has been attempted in two ways.

The *first* of these is characterized by Hilbert's famous monograph on the foundations of geometry (Göttingen, 1899). The idea is simple enough. Intuition, as shown by the history of the parallel postulate, cannot give certainty. Hence it is necessary to eliminate it, as far as possible. This can be done by examining the entire system carefully, collecting all intuitive

elements, reducing these to a minimum, and placing them at the *beginning* of the work in the form of axioms and postulates. The vulnerable area of the system is thus confined to the foundations. The foundations being granted, the system itself may be called logically perfect. As Hilbert puts it:

"'Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis *of our intuition of space.*'

"'The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent* axioms, and to deduce from these the most important geometric theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.'"

Another peculiarity of this method consists in assuming the existence of certain indefinables. Hilbert's indefinables are points, straight lines and planes. Between these indefinables certain relations are said to exist (*e. g.,* "between," "congruent," "parallel," etc.). These relations are undefined, in the usual sense of the terms. But their meaning is made clear—and so they really are defined, by a group of axioms. The meaning of the word "between" in geometry, for instance, is brought out by the "axioms of order." Hilbert has five groups of axioms. "Each of these groups," he says, "expresses, by itself, certain related fundamental facts of our *intuition.*" He names these groups as follows:

I. Axioms of connection (7 in number).
II. Axioms of order (5 in number).
III. Axiom of parallels (Euclid's axiom).
IV. Axioms of congruence (6 in number).
V. Axiom of continuity (Axiom of Archimedes).

With these *twenty* axioms, according to Hilbert, a rigorous geometry can be built up. "Hilbert followed essentially the Euclidean method, with a wonderfully keen logic. And yet certain defects were pointed out by Moore and others, showing

how difficult it is to satisfy logic when one seeks to determine the fundamental operations by the intuitional method" (Withers).

This partial breakdown of Hilbert's system has served to strengthen the *second method* of developing a rigorous geometry. It is the method of *arithmetization,* as Professor Pierpont calls it. This is considered by many modern writers as the only *rigorous* method. It consists in representing not only the geometric elements (*e. g.*, points, lines, surfaces) by (algebraic) symbols, but even the relations and operations connecting them. Figures are thus eliminated entirely, and geometry becomes a branch of symbolic logic.

In this connection may be mentioned the "geometric calculus" of G. Peano (German translation, Leipzig, 1891). "This calculus," says Peano, "which was anticipated by Leibnitz, has been developed, in this century, in various forms, especially by Möbius (1827), Bellavitis (1832), Grassmann (1844) and Hamilton (1853)." The entire treatise contains but one figure.

An able defense of this symbolic notation (Begriffsschrift) is given by Mr. H. Dingler in the "Jahresbericht der deutschen Mathematikervereinigung" of 1905 (pp. 581–584). He says: "If we have a number of concepts *A, B,* . . . , which are mutually connected in the manner indicated by the axioms, then these concepts are governed by all laws logically deduced in our subsequent development. Now, this structure evidently is no longer a geometry, but a branch of the science of pure logic. We observe that this purely logical structure differs from the previous geometrical-logical structure *only in a change of nomenclature* (Bezeichnung). It thus appears that the two systems, logically considered, are identical, so that we might say conversely: If we have certain concepts *A, B,* . . . obeying the given axioms, we may substitute for these, in the logical edifice constructed with the axioms, the concepts straight line, point, etc., and obtain as a result Euclidean geometry.

Hence we may say: If mathematics defines its notions through axioms [symbolically], it differs only in nomenclature from pure logic." The chief value of this procedure is the resulting economy. For, "if in a logical structure we replace the concepts *B* by the corresponding concept *A,* we have at once the

entire science that can be built on the concepts *A*." This is illustrated by the far-reaching principle of duality.

In concluding this first part of our subject, then, it may be said that to-day mathematics is *arithmetized*. The mathematician has absolute confidence in the intuition of pure number. Says Poincaré: "In the analysis of to-day, when one cares to take the trouble to be rigorous, there can be nothing but syllogisms or appeals to this intuition of pure number, the *only intuition which can not deceive us*. It may be said that to-day absolute rigor is attained."

The contrast between the old and the new is typified beautifully by two great thinkers: Plato and Gauss.

Plato declared that the divine thinking was geometric. "God always geometrizes." (Plutarch: *Πλάτων ἔλεγε τόν θεὸν ἀεὶ γεωμετρεῖν*). The same Plato defined geometry as the science of eternal truth (*τοῦ γὰρ ἀεὶ ὄντος· ἡ γεωμετρικὴ γνῶσις ἐστίν. Πολιτεία*, VII, 5276). To Plato, moreover, we owe the famous line: *Μηδεὶς ἀγεωμέτρητος εἰσίτω μου τήν στέγην.*

C. Fr. Gauss, the "princeps mathematicorum," said on his death-bed: *Ὁ θεὸς ἀριθμητίζει.* (Divine thinking is *arithmetic*).

II. Our second question is: Can the school take the attitude of the rigorous mathematician? In other words, should the scientific (symbolic) method be followed to the extent of reducing intuition to a minimum, and of using a set of independent axioms?

The answer is, most emphatically, No! On the contrary, that a careful cultivation of the intuitional powers is of the utmost importance, and that a rigid scientific method is out of the question in the school, is the practically unanimous opinion of representative mathematicians, both investigators and teachers.

In corroboration of this statement may be quoted the views of such men as E. H. Moore, F. Klein, Poincaré, Mach, Killing, Veronese, Simon, Holzmüller and many others.

Professor E. H. Moore's remarkable address "On the Foundations of Mathematics" (December 29, 1902), after referring to the redundancy of Hilbert's indefinables, contains these significant passages:

"The compatibility and the independence of the postulates

of a system of postulates of a special deductive science have been up to this time always made to depend upon the self-consistency of some other deductive science; for instance, geometry depends thus upon analysis, or analysis upon geometry. The fundamental and still unsolved problem in this direction is that of the direct proof of the compatibility of the postulates of arithmetic, or of the real number system of analysis.

"The Italian writers on abstract mathematics for the most part make use of Peano's symbolism. One may be tempted to feel that this symbolism is not an essential part of their work. . . . But of course the essential difficulties are not to be obviated by the use of any symbolism, however delicate.

"Indeed, the question arises whether the abstract mathematicians in making precise the metes and bounds of logic and the special deductive sciences are not losing sight of the evolutionary character of all life-processes, whether in the individual or in the race. Certainly the logicians do not consider their science as something now fixed. All science, logic and mathematics included, is a function of the epoch—all science in its ideals as well as in its achievements. Thus with Hilbert let a special deductive or mathematical science be based upon a finite number of symbols related by a finite number of compatible postulates, every proposition of the science being deducible by a finite number of logical steps from the postulates. The content of this conception is far from absolute. It involves what presuppositions as to general logic? What is a finite number? In what sense is a postulate—for example, that any two distinct points determine a line—a single postulate? What are the permissible logical steps of deduction? Would the usual syllogistic steps of formal logic suffice? Would they suffice even with the aid of the principle of mathematical induction, in which Poincaré finds the essential synthetic element of mathematical argumentation, the basis of that generality without which there would be no science? In what sense is mathematical induction a single logical step of deduction?

"One has, then, the feeling that the carrying out in the absolute sense of the program of the abstract mathematician will be found impossible."

Professor Moore says that the usual classification of the sciences into mathematical (deductive) and natural (inductive)

sciences may be satisfactory as an ideal one, but that "it fails to recognize the fact that in mathematical research one by no means confines himself to processes which are mathematical according to this definition; and if this is true with respect to the research of professional mathematicians, how much more is it true with respect to the study, which should throughout be conducted in the spirit of research, on the part of students of mathematics in the elementary schools and colleges and universities?"

With regard to the facts of mathematics Professor Moore says that the formulas of science are of the nature of more or less exact *descriptions* of phenomena; they are not of the nature of *explanations*. He insists that much concrete work be done at the beginning of the course, that the emphasis be placed on the *comprehension* of propositions rather than upon the *exhibition* of comprehension, that the *desire* for the formal proof be awakened before the formal proof itself is developed. "The teacher would do well not to undertake to make the system of axioms thoroughly complete in the abstract sense." The study of abstract geometry should be reserved, he thinks, for the later collegiate and university years.*

M. H. Poincaré has done much to make clear the place of intuition in mathematics. Himself a rigorist, he denies that certainty can come from intuition. And yet he attributes to it an absolutely indispensable rôle in these eloquent words:

"In becoming rigorous, mathematical science takes a character so artificial as to strike every one; it forgets its historical origins; we see how the questions can be answered, we no longer see how and why they are put.

"'This shows us that *logic is not enough;* that the science of demonstration is not all science and that intuition must retain its rôle as complement, I was about to say, as counterpoise or as antidote of logic.

"'I have already had occasion to insist on the place intuition should hold in the teaching of the mathematical sciences. Without it young minds could not make a beginning in the understanding of mathematics; they could not learn to love it and would see in it only a vain logomachy; above all, without intui-

* *School Review,* June, 1903.

tion they would never become capable of *applying* mathematics.'"

Even the creative scientist, however, according to Poincaré, depends on intuition.

Poincaré says that logic guarantees the validity of each step in the demonstration, but that we have to have, besides, an instrument by which we may determine the *unity* of the demonstration. "We need a faculty which makes us see the end from afar, and intuition is this faculty. It is necessary to the explorer for choosing his route; it is not less so to the one following his trail who wants to know why he chose it." "Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we put in their place. . . . In mathematics logic is called *analysis,* and analysis means division, dissection. . . . Thus logic and intuition have each their necessary rôle. Each is indispensable. Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention."*

Similar remarks may be found in *Mach's* work, Euclid's systematic treatment being criticized in these words:

"Through this endeavor to support every notion by another, and to learn to direct knowledge the least possible scope, geometry was gradually detached from the empirical soil out of which it had sprung. People accustomed themselves to regard the derived truths as of higher dignity than the directly perceived truths, and ultimately came to demand proofs for propositions which no one ever seriously doubted. Thus arose—as tradition would have it, to check the onslaught of the Sophists—the system of Euclid with its logical perfection and finish. Yet not only were the ways of research designedly concealed by this artificial method of stringing propositions on an arbitrarily chosen thread of deduction, but the varied organic connection between the principles of geometry was quite lost sight of. This system was more fitted to produce narrow-minded and sterile pedants than fruitful, productive investigators." And in a supplementary note he says: "Science is not a feat of legal casuistry. Scientific presentation aims so to expound all the grounds of an idea that it can at any time be thoroughly exam-

* "The Value of Science," *Pop. Sc. Monthly,* Sept., 1906.

ined as to its tenability and power. The learner is not to be led half-blindfolded" (p. 113).

H. Hankel, the historian, contrasting Euclid's work with the geometry of the Hindus, says: "The peculiar advantages of the Greed method and of the Hindu method might be blended, by sharply defining the fundamental principles and placing these at the beginning of the subject, and then following the lead of the Hindus. In that case Euclid's geometric conglomerate could be transformed, without question, into a system characterized by a progressive arrangement, not of accidental, but of essential, ideas. If geometric instruction were based on such a system, the pupil would in reality derive that profit which he is supposed to derive, while at present he anxiously clings to trivial laws of congruence and similarity, but rarely attains to an independent geometric intuition [Anschauung]."*

Killing freely concedes that it seems to be a mistake to banish motion from geometry. He intimates that Euclid's frantic efforts to avoid superposition lead to an artificial system. The fact that Euclid employs motion only in I, 4, I, 8, and III, 24, caused Professor Simon† to question the genuineness of these proofs. But even if that were true, it would not save the situation.

Veronese does not hesitate to say that the school needs a larger body of axioms than abstract science. This, of course, means an unconditional surrender of the principle of absolute rigor.

We have seen, then, that for psychological, historical and even mathematical reasons we must attribute to intuition an indispensable place in mathematics. If the same question is examined from a purely pedagogic standpoint, the case is still stronger in favor of intuition.

Professor F. Klein's views on the subject are well known. For a representative account of these I may refer to a collection of lectures: "Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an den höheren Schulen, F. Klein und E. Riecke, Leipzig, 1904." Some of these lectures were summarized by Professor J. W. A. Young in *Bull. Am. Math. Society* (April, 1906). Klein's definition of elementary

* "Geschichte der Mathematik," p. 208.

† "Euclid und die sechs planimetrischen Bücher," p. 44.

mathematics is significant. "The only definition of elementary mathematics that will be of any use to the school must be a practical one: In all domains of mathematics those parts are to be called elementary which can be understood by a pupil of average ability without long continued special study." Of course, logical training should not be undervalued, but the most important task of mathematical instruction consists in the strengthening of space intuition and in training to the habit of functional thinking.

To support his position, Klein quotes approvingly analogous statements of his colleagues in Germany and France, notably of Holzmüller and the brothers Tannery.

G. Holzmüller, a prominent teacher and successful text-book writer, protests against the excessive arithmetization of secondary mathematics. He says that the foundations of geometry are a suitable subject for specialists like Veronese, Killing, Hilbert, but that even the specialists have not, thus far, answered all pertinent questions satisfactorily. "The school must not aim at absolute rigor nor at a flawless system. It must be satisfied with a methodic selection."*

In their remarkable book, "Notions de Mathématiques" and "Notions Historiques," Jules Tannery and Paul Tannery use intuition freely. The book is based on the official programme of May, 1902. The following passage is especially interesting: "Le professeur laissera de côté toutes les questions subtiles que soulève une exposition rigoureuse de la théorie des dérivées; il aura surtout en vue les applications et ne craindra pas de faire appel à l'intuition."

The official instructions of July, 1905, in regard to geometry teaching are extremely suggestive. It is urged that teachers begin the subject concretely, that there be constant appeals to the pupil's experience, that all statements which are self-evident to the pupil be tested as experimental truths, that the constant use of motion is of the greatest advantage, etc.† Similar passages might be quoted from the Prussian and Italian regulations.

Of course, it must be remembered that geometry is begun from two to three years earlier in European schools. Hence,

* *Zeitschrift für mathematischen und naturwissensch. Unterricht*, 1902.

† Bourlet's "Géométrie," Paris, 1907.

the constant insistence on the concrete side is justified. But even for a mature student these considerations are valid. In regard to the use of motion, for instance, Mach says: "The study of the movement of rigid bodies, which Euclid studiously avoids and only covertly introduces in his principle of congruence, is to this day the device best adapted to elementary instruction in geometry. An idea is best made the possession of the learner by the method by which it has been found" (*l. c.*, p. 112). And Moore says: "Indeed, one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of science in the modern sense might not have been so long deferred."

It is a matter of congratulation that all these considerations have finally brought about a healthy reform movement. The most progressive teachers are finally beginning to realize that it is useless to offer a pupil material which he cannot possibly digest, or which—even if absorbed—leads to no permanent benefit. In Europe this feeling has led to the organization of preliminary or "propædeutic" courses extending over a period of one or two years. Able discussions of the merits of such a course may be found in Reidt's "Anleitung zum mathematischen Unterricht" (Berlin, 1906), in Schotters's "Inhalt und Methode des planimetrischen Unterrichts," and in Simon's "Methodik des Rechnens und der Mathematik." For many years such work has been in progress in the German elementary and secondary schools. Nor is this true of the German schools alone. In France and Italy considerable preliminary work is now done before the pupil begins his demonstrative geometry. Dozens of European propædeutic geometries are now available. Many of these are excellent. Some are written by prominent mathematicians, *e. g.*, those of Veronese,* Holzmüller, Simon, etc. Of these, Professor Simon's is extremely interesting. In a letter addressed to Professor Klein, a friend and class-mate of his, he gives a very complete account of his own class-room experience in this field. After a few introductory paragraphs he submits a detailed outline of a large part of the course. I will give only the introduction: "During your visit in Strassburg you showed interest in the first instruction in geometry,

* "Nozioni elementari di geometria intuitiva," Verona, 1906.

given in the fourth class [boys of twelve], for which about 86 hours, at two periods a week, are at our disposal. You were surprised that in spite of being so decidedly systematic in my writings I could be so thoroughly undogmatic in my teaching, and you desired an explanation. At this propædeutic stage it is my principle to emphasize most sharply the historical, *i. e.*, the experimental, origin of elementary geometry, to emphasize untiringly its connections with life, surveying, architecture, ceramics, machinery. Every circle is a wheel, every tangent a belt. *I depend as much as possible on intuition.* The eye is the most important aid of geometry, and—along with it—motion. This is in the sharpest possible contrast to that new scholasticism as embodied especially in the excellent Italian texts of Ingrami and Veronese and as it has found its typical German exponent in Hilbert. We agreed in the belief that this new scholasticism—even if its scientific value be duly recognized—as soon as it becomes the method of instruction, endangers the valuation of mathematics from the standpoint of life, and that, by giving one-sided attention to the logical aspect of mathematics, it dwarfs all other aspects and so robs mathematics of its central position in the natural and technical sciences." In regard to the early demonstrative work he says: "At the beginning I pay no attention whatever to proofs, but am satisfied with the 'Behold!' of the Hindus, until by very slow degrees I have brought my pupils to the point of searching for reasons."*

Throughout this development the English, in their usual conservative way, have been following their own course. More than two centuries ago their famous countryman, John Locke, had written eminently sensible things about education. He had pronounced intuitive knowledge irresistible, comparing it to the direct, bright sunlight. He had advised that obscure things be approached by gentle and regular steps and that what is most visible, easy and obvious in them he considered first. He had eulogized mathematics which, he thought, "should be taught all those who have the time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures." Furthermore, the 'Inventional Geometry' of William George Spencer, the father of Herbert Spencer, was written more than a generation ago. In spite of this, Euclid reigned

* *Jahresbericht der deutsch. Math. Ver.*, 1904.

supreme. Very young pupils were made to memorize the classical proofs verbatim. At last the reaction came, and, as might have been expected, in a more violent form than in any other country. It came in the form of the Perry movement. From a slavish adherence to the old Greek master the followers of Perry have swung to the opposite extreme: *A good teacher will occasionally demonstrate a theorem. Everything is to be observational, experimental.* R. Fricke has given an admirable report of the Perry movement in the *Jahresbericht d. d. Math. Ver.*, 1904. Perry wishes to adapt mathematics to the average mind and expects human nature to overcome all natural difficulties. He says: "So now we teach all boys what is called mathematical philosophy, that we may catch in our net the one demigod, the one pure mathematician, and we do our best to miss all the others."*

Whatever one may think of the Perry movement, of laboratory method and experimental devices, etc., there can be no question that a very determined effort must be made to find out what the learner *really comprehends,* how much of the subject-matter he retains, and what can be done to invest the subject with greater interest. For it is undeniable that there prevails in the educational world of to-day an indifference to the importance of mathematics "so alarming and so widespread as to call for an active campaign on the part of those who wish to see this branch of education accomplish results commensurate with its importance and its possibilities." (See Report of Assoc. Math. Teachers New Engl., Boston, 1906.) This indifference is not confined to the educational world. It is very general. "The teacher who is a specialist in mathematics is regarded to-day by the general public as a monstrous phenomenon or a crank." Who is not reminded at once of Oliver Wendell Holmes' biting satire on the mere mathematician: "I have an immense respect for a man of talents *plus* 'the mathematics.' But the calculating power alone should seem to be the least human of qualities, and to have the smallest amount of reason in it; since a machine can be made to do the work of three of four calculators, and better than any one of them. Sometimes I have been troubled that I have not a deeper intuitive apprehension of the relations of numbers. But the triumph of the ciphering hand-organ has

* "Report of British Association meeting," Glasgow, 1901.

consoled me. I always fancy I can hear the wheels clicking in a calculator's brain. The power of dealing with numbers is a kind of 'detached lever' arrangement, which may be put into a mighty poor watch. I suppose it is about as common as the power of moving the ears voluntarily, which is a moderately rare endowment."

Schopenhauer rails at mathematics in a similar strain. He calls Euclid's proofs "mouse-trap" demonstrations. Concerning arithmetic he says: "That the lowest of all mental activities is the arithmetic, is proved by the fact that it is the only one which can be executed also by a machine."

How absurd all such arguments are has been shown admirably by Pringsheim,* who refers to a machine constructed by Stanley Jevons by which certain logical conclusions can be obtained in a purely mechanical way. Accordingly, he says, all logical thinking is degraded to the level of arithmetic thinking (in Schopenhauer's sense). But all reasoning must be logical. Hence, reasoning is a low mental activity.

Of course, one might oppose to these criticisms the old story concerning the royal road to learning. One might even point out the brilliant line of great men who excelled in mathematics, men prominent in all departments of human thought and endeavor. And that mathematics is not necessarily dry, might be proved by the utterances of the most fanciful of all creatures—of great poets, even if these be not uniformly so romantic as the German poet Novalis, when he writes: "The life of the gods is mathematics. All divine messengers must be mathematicians. Pure mathematics is religion. Mathematicians are the only happy creatures. The mathematicians know everything, etc."

We may safely conclude from all this that the school cannot follow the program laid down by the rigorist. Its procedure must be not only logical but also psychological. It must appeal to the interest of the learner and must lead him into the subject by slow steps adapted to his powers of comprehension. Only then will it succeed in making clear to him the tremendous and absolutely unique importance of mathematics.

III. There remains, now, one final question. Precisely which element has intuition contributed to the construction of geometry, and to what extent is it operating to-day in the class-room?

* *Jahresb. d. d. Math. Ver.*, 1904.

The first part of this question can be answered only by the historian and the second part by the psychologist and logician.

Fortunately, owing to the wonderful revival of interest in historical studies, it is now possible to answer many historical questions with reasonable accuracy. For a preliminary account I may refer to Professor Cajori's address on the "Lessons Drawn from the History of Science."* It is shown how slow the evolution of scientific concepts is. Concerning the teaching of geometry, says Cajori, "history tells us to rely greatly upon intuition in the early stages, and to make no sudden and abrupt change to severely rational demonstration. Modern critics warn readers to take no notice of geometric figures or diagrams. Among them, geometry without diagrams is the order of the day. This is the very highest step in the echelon of geometric abstraction. In my judgment it should not be attempted at all in secondary education, but should be postponed until the advanced college and graduate coures. . . . The historic and, I believe, the pedagogic method is to start at the bottom with sense-perception, with the 'art of handling the rule and compasses,' and then gradually to rise to higher levels of abstraction. . . . There are cases on record where great men have resorted not only to intuitional, but also to experimental mathematics."

How true these remarks are, a mere glance at the history of the subject will show. In the first place, nature itself offers a multitude of incentives to the study of geometry. The concepts of point, line, surface, solid, angle, are suggested by thousands of natural objects. Thus the idea of a point might be caused by the stars, by grains of sand, or particles of dust floating in the sunlight, by drops of water. The trees, blades of grass, the arms and fingers, the rainbow, spiderwebs, the rays of the sun, the path of a falling body, the march of the heavenly bodies, and a multitude of other objects and observations lead to the idea of a line. A surface is suggested by any expanse of water, a sheet of ice, a level or rolling section of country, by the palm of the hand, by the contours of all objects, by the sky. Finally, the infinite variety of the natural objects surrounding us almost compels us to make an analysis of form. We are thus led, early in life, to acquire a knowledge of the principal

* *School Science and Mathematics,* February, 1908.

geometric forms. Nature, in other words, furnishes the raw material of geometry.*

It could easily be shown that this is not merely a theory. Concerning the overpowering influence of astronomical observations, for example, Troels-Lund has written most eloquently.† The most favorite architectural form with many Semitic tribes was the cupola, the conscious reproduction of the starlit evening sky. Mach says: "It is extremely probable that the experiences of the visual sense were the cause of the rapidity with which geometry developed."

And yet, there is a great difference between accidental observations, however numerous, and their conscious apperception or even application. As Mach puts it: "Rays of light in dust or smoke-laden air furnish admirable *visualizations* of straight lines. But we can derive the metrical properties of straight lines from rays of light just as little as we can derive them from *imaged* straight lines. For this purpose experiences with *physical* objects are absolutely necessary. The *rope-stretching* of the practical geometers is certainly older than the use of the theodolite. But once knowing the physical straight line, the ray of light furnishes a very distinct and handy means of reaching new points of view. A blind man could scarcely have invented modern synthetic geometry."

What gave our forefathers this essential experience with physical objects? The answer is simple. The practical necessities of life compelled them to make the tremendous transition from an aimless, almost unconscious contemplation of nature to its purposive mastery; the necessity of securing shelter, clothing, food. It would be extremely interesting to show how the shelter of primitive man was but an attempt to reproduce artificially the natural shelter provided by the forest or the caves. This might be shown of the tepee of the Indian, the round hut of the South-African negro, the log house of the forest dweller. We see here the conscious apperception of the cone, the cylinder, the sphere, the rectangular block. Then came the era of stone buildings. The Cyclopean walls and the pyramids with their slanting walls show the difficulty encountered in the

* Eberhard, "Über die Grundlagen und Ziele der Raumlehre."

† "Himmelsbild und Weltanschauung im Wandel der Zeiten," German transl., Leipzig, 1900.

erection of vertical buildings. That was possible only after the discovery of the right angle, of vertical and horizontal lines. Then came the use of rectangular building blocks—the most convenient form for securing vertical and horizontal lines. The first instruments were invented, rulers and compasses (derivation of *circumference*) and, possibly, try-squares and plumb bobs. These building activities, then, firmly established certain geometric forms, such as rectangles, triangles, circles, and necessitated the use of tools.*

Not quite so productive geometrically, and yet very important in some respects, were the efforts of primitive man to secure suitable clothing. These finally led to the invention of the art of weaving, an art practiced by all savage tribes. To it we owe an intimate knowledge of parallels, of parallelograms and triangles.†

Above all, it was necessary for primitive man to obtain food and to store it. This led to the invention of weapons, and of household implements, and ultimately compelled the adoption of agricultural pursuits. Much might be said of the geometric significance of the weapons, the bows, arrows and spears; of the early household utensils, such as knives, spoons, vases. All that is relatively unimportant when we consider the tremendous influence of agriculture on the development of geometry. The very word "geometry" exhales the odor of the soil. All the early writers, notably Herodotus, tell us that geometry originated in Egypt, where the annual overflowing of the Nile made periodic surveys necessary. We know that in the early agricultural settlements land was divided equally among the settlers, usually in the form of rectangles and squares.‡ In this manner the first rules of mensuration were obtained empirically. That ancient Babylon was not ignorant of these, was made clear by the recent researches of Hilprecht.§ The names given to our units of length still remind us of this first awakening of geometric reasoning. (Cf. foot, span, cubit, ell, pace, mile.)

* Cf. Hankel, p. 72. Also, Lübke-Semran, "Die Kunst des Altertums," Esslingen, 1908, pp. 1–10.

† Cf. Mach, p. 55. Also, Dr. E. Wilk, "Der gegennivärtige Stand der Geometrie-Methodik," Dresden, 1901, p. 25.

‡ Cf. Hankel, p. 78, and p. 82.

§ See *Bull. Am. Math. Soc.*, Vol. XIII, 1907, p. 392.

This rapid survey of the sources of geometry would not be complete without a reference to the æsthetic impulses of man as shown by his constant habit of ornamenting even the commonest articles. He decorates his body (tattooing), his garments, his weapons, his pottery, his houses. A wealth of information concerning the geometric notions of primitive man could be obtained simply by a study of his ornaments. The importance of this field is only beginning to be appreciated. Cantor refers to it briefly (p. 58), giving a few geometric designs and suggesting that the professional mathematician among the Egyptians might have been stimulated by them to investigate their properties. He also speaks of the practice of the Egyptians to rule their walls into equal squares for drawing purposes, and sees in this the first dawn of a theory of proportions and similarity. Very valuable material may be found in A. C. Haddon's "Evolution in Art" (London, 1895), in L. Frobenius's "Aus den Flegeljahren der Menschheit" (Hannover, 1901), and in Owen Jones's "Grammar of Ornament." Two German teachers, J. Jahne and Hans Barbisch, have systematically introduced historic designs into their early course in geometry. In the laying of tile floors such as have been found in Pompeii and Olympia, we may see the occasion for the discovery of numerous geometric theorems. It can hardly be doubted that the innate desire to ornament led to the conscious use of the principles of symmetry, congruence, equality, similarity. Among the theorems that may have originated in this way is that concerning the sum of the angles of a triangle. Says Mach: "This fact could not possibly have escaped the clay and stone workers of Assyria, Egypt, Greece, etc., in constructing their mosaics and pavements from differently colored stones of the same shape. The theorem of the Pythagoreans that the plane space about a point can be completely filled by only three regular polygons, viz., by six equilateral triangles, by four squares, and by three regular hexagons, points to the same source" (p. 55).

We may summarize these results as follows:

1. We owe to observation, i. e., to experience, our fundamental geometric concepts. "The choice of the concepts is suggested by the facts. . . . The importance of the concepts is estimated by their range of application. This is why the con-

cepts of the straight line and the plane are placed in the foreground, for every geometrical object can be split up with sufficient approximateness into elements bounded by straight lines and planes" (Mach, p. 86).

2. The connections existing between these concepts are a matter of intuition, i. e., our so-called axioms of order, of connection, of parallels, congruence, and continuity are empirical (Hölder, p. 21).

3. The arrangement of theorems is determined by experience (Mach, p. 91). "From this point of view we understand at once the *form* geometry has assumed—the emphasis, for example, that it lays upon its propositions concerning triangles. . . . For geometry is obliged, both in its own interests and in its rôle as an auxiliary science, as well as in the pursuit of practical ends, to answer questions that *recur repeatedly in the same form*. Hence it is desirable to collect the most general possible propositions having the widest range of application."

We have thus seen how gradual was the ascent from intuition to logic. In fact, so tremendous is the effect of the historical development that gave rise to geometry that "the historical influences of physiological space in the development of the concepts of geometric space cannot be eliminated." Poincaré uses this beautiful comparison: "You have doubtless seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place."

If this can be the view of a professional mathematician, how much more should it be constantly borne in mind by the teacher in the instruction of the young. In laying the foundation of the subject, he should follow the experience of the race. The same raw material is at hand everywhere. The force of gravity once led to the discovery of vertical and horizontal lines. It still determines the form of our houses. It still explains the

constant recurrence of right angles, parallel lines, rectangles and triangles. A thousand observations are constantly available to secure the conscious recognition of geometric forms. The connections between these forms are still best developed by the use of the simplest drawing instruments. And the first theorems are still best apperceived intuitively.

This brings us to the last difficult query: Where does intuition stop and logic begin?

The answer is two-fold. The rigorist lays a more or less intuitive foundation and then uses logic exclusively. The school uses intuition even in the demonstrative work. For the school will never use a geometry without figures. Every reference to a figure, however, is an appeal to intuition. It is for this reason that the rigorists have devised the symbolic method. "Symbolic representation has the disadvantage that the object represented is very easily lost sight of, and that operations are continued with the symbols to which frequently no object whatever corresponds." In other words, the symbolist can guarantee the logical accuracy of his investigations, but he is not sure whether his results have a significance in the material universe. He can imagine a universe in which his particular results might hold. But he cannot tell whether the actually existing world satisfies his solutions. The school geometry, on the other hand, by constantly referring to diagrams, never loses connection with the physical world. In this mixture of logic and intuition lies its strength and its weakness. Its strength, in that the pupil can more readily follow a semi-intuitive demonstration; its weakness, in that intuition cannot give absolute certainty.

We have now the key to the solution of the old puzzle: Why are the results of a geometric investigation in agreement with the facts of the external world, even though we seem to pay not the slightest attention to the external world during a demonstration? The entire question depends on a misconception. Whenever we use a figure, we are as a matter of fact in constant communication with the physical world and our intuition does not allow us to take a step that is not in agreement with that world. This is the essence of *Kroman's theory*. Kroman thinks that whenever we study a special figure, "we are able by rapid variations to impart all possible forms to the figure in thought

and so convince ourselves of the admissibility of the same mode of inference in all special cases." There is an historic basis for this opinion. We know that even the Greeks at first proved many special cases before they announced the general theorem [Hankel, p. 96]. In proving a proposition by means of a figure, we have no absolute rule of procedure. We are usually obliged to introduce construction lines. We cannot predict in advance the effect of these construction lines, but are obliged to feel our way and search for conclusions. Thus we are performing an experiment on the basis of which we finally state a certain conclusion. In other words, we have substituted an "experiment in thought for a real experiment, and that is what we mean by deduction" (Hölder, p. 10). "The *geometric* mental experiment has advantage over the physical only in the respect that it can be performed with far simpler experiences and with such as have been more easily and almost unconsciously acquired" (Mach, p. 87). If this view is correct, and that seems to be the case, it is wrong *to call ordinary geometry a purely deductive science.* As a matter of fact, whenever we use a figure, induction and deduction go hand in hand, precisely as in an experiment in physics. Every construction line represents a tool with which the learner must become familiar through experience; a tool the power of which he cannot possibly anticipate. This explains perfectly why so many students are confused and bewildered when they come into contact with demonstrative geometry, why they even question their own power of reasoning. The fact is that the geometry of the school is not purely a matter of reasoning, but that it requires a certain experimental skill that can come only through experience.

It will facilitate the pedagogic process greatly if teachers will constantly bear in mind that they must obtain intuitively the fundamental concepts, the axioms and many elementary theorems, and that intuition enters into every demonstration employing a figure.

In conclusion, it may be said that geometry may be studied according to the spirit of two different methods. Rational or abstract geometry limits intuition to the foundation and erects on this a purely logical structure irrespective of external realities. The geometry of the school is a MIXTURE of rational and

intuitive geomery. It is semi-experimental. On an intuitive basis it erects a logical edifice under the constant supervision of the architect Intuition.

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SOME THOUGHTS ON SPACE.

By E. D. Roe, Jr.

The statement is frequently made that Kant's theory of space has been overthrown.

The question at issue is only the old battle as to whether all our knowledge is derived from sensation or not,* renewed on the field of geometry. After an armistice, the fortifications built by Kant have been bombarded by the modern heavy and light artillery and rapid fire guns of empirical psychology, and while some defects have been disclosed in the walls, it is pertinent to look and see after the smoke of battle has cleared, if the walls are razed to the ground.

Doubtless to the extreme to which Kant identified his theory with the Euclidean form of space a correction needs to be made. But the question still arises, would this be the case without this identification, and is his view overthrown *in toto?*

It is the object of this communication to raise this question, and ask if any modification of his theory can be proposed which leaves any part of it still in force. It has seemed to the writer that the following considerations might be suggested.

The fundamental principle of Kant was that whatever experience† occurs (to a finite reason) must occur in time, and,

* The way in which Leibnitz touched up Aristotle on this question is as good to day as ever. Says Leibnitz: "Hence, to the well-known adage of Aristotle, 'nihil est in intellectu quod non fuit prius in sensu,' I have added this qualification—nisi intellectus ipse," an exception which is almost the whole thing, and which shows that this seven-headed thinker (Leibnitz) who preceded Kant anticipated the Kantian theory as he did many another discovery, for his exception to or revision of Aristotle is the Kantian theory in a nut shell.

† What Kant means by experience is a concatenated experience. A mere experience, or stream of sensations could be had in a consciousness without the categories and intuitions. They would not be necessary as a condition of its possibility. But this is not the kind of

if external, must occur in time and space. Time and space are necessary conditions of the possibility of any experience whatever. It would seem that these principles are not overthrown, nor can they be overthrown, least of all by experience, for which they are the condition of possibility. Thus it can be said prior to or before the event, if the sun rises tomorrow morning, it must rise in time and space.

But Kant went farther and held that not only is space in general the necessary condition for the possibility of the experience of external objects, but indeed a Euclidean space. If the sun rises tomorrow morning it must rise in time and in a Euclidean space.

His error then consisted in identifying the general form of space with a particular form, the Euclidean. If he had not tried to state the particular form as necessary, it seems that he would have been so far unassailable.

The question next arises why do we assume the Euclidean when the mind has at least two other forms which would account for the facts equally well. Possibly experience has had some influence, but first of all experience has never shown us and probably never can show us that we are in a Euclidean space. How then has it influenced us in a matter of which it is acknowledged that it is impotent to be the arbiter? Is there anything else which has influenced us? It would seem that reason in its unconscious but correct working has influenced us.

The Euclidean form is the simplest form which the mind has, and in the course of the mind's childhood, was the easier to grasp and apply. By maturer reflection it was discovered by a process of pure reasoning, not by experiments on nature, that the mind had other forms of space, of which two at least, the hyperbolic and the elliptic forms, would, with proper attention to the parameters have applied to our space. In fact all three approach the same limit, as the parameter becomes indefinitely great. And all three may be regarded as special forms of a general form which includes in itself all three. In not applying the hyperbolic and elliptic forms, the mind unconsciously and

experience which the theory of knowledge is called upon to explain. The categories and forms of intuition are necessary to the possibility of such a rational and rationally connected experience as we find ours in consciousness to be.

correctly followed a logical demand (of its own creation) that of alternative or possible hypotheses to explain an experience the simplest is to be chosen.

Thus we have not derived the knowledge of other forms than the Euclidean from experience, and as these and the Euclidean can be regarded as particular forms of a general form, if we say Euclidean space is derived from experience, we have to explain how it is that the particular is, while the general which contains it and presupposes it is not, derived from experience. Moreover, all experiments conducted with a view to determine whether space is derived from experience are vitiated by the fact that the exact knowledge of space has to be presupposed before the experiment, in other words the experiment is superfluous.

That Kant posited the Euclidean form as necessary was an inadvertance for which he is not to be severely blamed, as the other forms had not been matured by reason in his day. His error is somewhat analogous to the errors made in analysis, where in old time infinite divergent series were used by many celebrated worthies. A sharper criticism showed the unsoundness of this use, but did not show that analysis was therefore of empirical origin. It simply showed that a mistake had been made and was corrected. *It vindicated analysis.* If the error in regard to space had been detected by empirical means, it would certainly have to be yielded that space is of empirical origin. But the very means used to detect the error show that several forms of space are at least created by the mind (they or some of them may also exist externally to it) and that among them as a special case is the Euclidean form. The exact perfection of all these forms shows that they are at least creatures of reason. Whether they have also objective existence is beyond a proof. Either their ideality or their objectivity lies in the land of belief, and the proper instrument for investigating this is the calculus of probabilities in its widest logical significance.

Compare their sharpness and exactness with any experimental or physiological fact. What idea at all have we of a color? None. It must be experienced, in order to be appreciated. But its effect is only in the sensation and with the sensation it is gone. What idea have we of weight? None. We can de-

termine its effects in time and space, to be sure. It is an intensive magnitude, and in so far as the assumed effects of intensive magnitudes appear in time and space, both the time and space appearance can be measured. But what is it itself? What idea have we of it?

In the case of a Euclidean triangle the idea is perfectly sharp. The conclusions from Euclidean premises are universal and necessary from the premises. Experience gives no such results. To say that, in demonstrating the theorem that the sum of the three interior angles of a Euclidean triangle is equal to two right angles, we are performing an experiment, is to trifle with words and misuse language. No experiment on nature is performed. No question is put to nature. Reason is only concerned in tracing out necessary connections between its own creations. It may be an experiment in a pedagogical sense whether a given student will prove the proposition, but this is a confusion of ideas and a throwing of dust in our faces in the place of argument. The real question is not whether this particular student will now prove it, *but can it be proved by reason.* And it is maintained here that this is the only way that it ever has been or ever can be proved. An experimental proof in the strict sense would be no proof at all. Nothing but a more or less close approximation would be obtained and obscurity and darkness would replace light.

In saying that time and space are *a priori* forms of intuition which are necessary to the possibility of experience, it is not meant that time and space are innate ideas (Descartes) fully developed in the mind before experience occurs, but only that when experience does occur and the mind is stimulated by something not of itself, it will act according to certain forms on the stimulus, and not otherwise, that is according to necessary forms of its own action. It can not act otherwise if it act at all. It possesses potentially the forms, but not actually before experience. And in its practical action only such as are practically necessary for its needs will be called out and applied. Thus the hyperbolic and elliptic forms were not needed for its practical ends and were only called out later by an introspective process. As far as time is concerned, the forms of space first appear at the time of experience. Experience is the occasion of their being brought out and applied. If no experience happened

they would never come forth. We might reverse the Kantian dictum and say experience is necessary to the possibility of the appearance in consciousness of the forms of time and space as well as of the categories. And Kant himself maintained this principle by implication.

The point however is that the forms of intuition and the categories were not something communicated to consciousness or handed over by experience. In fact the possibility of this communication would be inexplicable, if we did not first presuppose an understanding capable of receiving the communication, that is capable of itself forming the very ideas supposed to be communicated. And if it can itself form the ideas, and does so on the occasion of experience, here again the communication is superfluous, and again the simplest hypothesis is not that a handing over to a blank, which is unintelligible, took place, but that reason by its own spontaneity, at least, acted according to the laws of its own activity, and applied its forms and categories to the matter of experience.

It seems to me that the empirical psychologists are always concerned with a refined investigation of the empirical occasions on which the form of space is called into being as a response by the mind to the external occasions. As to these empirical occasions there is no necessary order. They may be highly variable in different cases, and they are of course empirical. But it is maintained the different empirical occasions are not space, nor the thought of it. They confuse these occasions with space, and thus believe that they have attached to space an empiricality which only belongs to the occasions on which the mind puts forth its form in consciousness.

If the mind is not constituted so to act space will not result no matter how much experience is had. A cow has the same empirical occasions, but does any one imagine that mathematical space is in a cow's mind, or ever will be, or that it could be revealed to or handed over to the cow, or by any instruction be conveyed to or gotten into the cow's consciousness? If experience could cause its genesis it would be there, for a cow has as much expereince as we, doubtless more, because the cow has nothing but experience and all of it in time and space.

A honey bee builds a comb which could only have been consciously deduced by an expert mathematician, but has the honey

bee derived it from reason or is it conscious of it? It would seem that here is not an innate idea but an innate instinct. The bee builds without being conscious of the complexity of the space relations or the complexity of the building. But if experience were the cause of the space concept, the honey bee, above all, should have it. But it gives no evidence of this, since for centuries it has built with dumb monotony the same invariable cell. Thus we see a mind is presupposed which is capable of itself forming the notion, in order that the notion may appear.

This does not prove that time and space are not objective realities also. Kant denied their objective reality. But he should not have done so as by his theory he did not know what was external to the mind. He should have neither denied nor asserted this. In fact so far as it is left to pure reason to decide, neither the ideality on the one hand, nor the objectivity on the other hand of time and space can be established or disestablished, but they are as we said before a concern of probability, a question for the practical reason.

In conclusion then, it does not seem that Kant's fundamental principle that time and space in general are necessary forms of intuition is overthrown, but that the mind acting by its own spontaneity applies its categories and forms of intuition to the matter of experience, and that in the way it has come to apply the Euclidean form it has not received a revelation from experience, nor can it probably receive such a one (the equally applicable hyperbolic and elliptic forms stand in the way of this) that our space is Euclidean, but rather has it chosen, according to its own laws and logical demands, to apply the simplest theory, the Euclidean theory, as entirely competent to its practical needs with reference to our space.

Finally our space may also be in fact Euclidean, and objectively exist, but probably this can neither be proved nor disproved. In fact if some measurements or observations in the visible universe would be better reconciled by a hyperbolic or elliptic space, such observations would demand most careful reinvestigation and adjustment. Still they would not prove that the mind has not these forms prior to their communication by experience, since we already have them, only we have not as yet discovered a practical need for their application.

Until such discoveries are made it is not scientific to use the thought of them for argument.

The preceding conclusion was reached independently. My attention has since been called to the book of Dr. Carus, "The Foundations of Geometry," Chicago, The Open Court Publishing Co., 1908. It is a great pleasure to recognize in him a friend and not an antagonist. The results here reached seem to agree with those of Dr. Carus, though the method and standpoint are a little different. What he calls "the apriority of motion" I should call one of the empirical occasions on which the mind by its spontaneity acts and brings into consciousness the form of space. It would seem that motion is not the only empirical occasion on which the form is called forth. Why might not one lie perfectly still with eyes closed and receive tactile sensations on different parts of his body and some notion of here and there be called out without the necessity of motion? I should replace his "abstraction" by spontaneity. He says space is not the product of pure reason. I should say it is, but not of pure reason in its discursive or logical activity by the application of the categories. That is I should maintain that pure reason has an activity outside of its logical action, and that this is not categorical action, but intuitive action. It is Kant's "Anschauung." That is pure reason has a faculty of pure thought and a faculty of pure intuition. This would merely be a question of definition of pure reason. These are some differences of view and statement which can not divide a house against itself.

If I were asked my reasons for preferring spontaneity to "abstractions," I should say: If the "abstractions" are abstractions from experience they are only general inductive or empirical laws and as such can never have the character of necessity or universality. This very theory has been offered by the opposite side, to explain away the universality and necessity claimed for the categories and forms of intuition, and I should not want to seem to favor it. Again to most persons I fear the word abstraction conveys an idea of unreality. "It is a mere abstraction." Therefore for the same reason that I would avoid the use of "imaginary" for "complex number," I would avoid a term which might cause a weakening of the thought to be

conveyed. For me spontaneity seems to be a positive word and to call up the thought of lively reality in action.

I also take pleasure in referring to a paper of Professor W. H. Metzler, "Geometry," *Secondary Education,* Bulletin 28, Albany, N. Y., 1905, pp. 47–54, in which he arrived at final results in harmony with those of Dr. Carus and the writer, that geometry is not an empirical physical science, but a pure science.

Syracuse University, May 27, 1909.

NOTES AND NEWS.

The Annual Meeting of the association will be held at the College of the City of New York on Saturday, November 27th. It is hoped that a large number of members will be present.

There is prospect of a very important contribution to the teaching of geometry, in the work of the recently organized National Geometry Syllabus Committee. This committee was appointed last spring as a joint committee of the National Educational Association, and the Federation of Teachers of the Mathematical and the Natural Sciences, and has now begun its work.

The committee is constituted as follows: Chairman, H. E. Slaught, The University of Chicago. (1) Sub-committee on Logical Considerations—Chairman, David Eugene Smith, Columbia University; Secretary, Eugene Randolph Smith, Polytechnic Preparatory School, Brooklyn; William Betz, East High School, Rochester, N. Y.; C. L. Bouton, Harvard University; W. B. Carpenter, Mechanics Arts High School, Boston, Mass. (2) Sub-committee on Lists of Basal Theorems: Chairman, E. R. Hedrick, University of Missouri; Secretary, W. W. Hart, Shortridge High School, Indianapolis, Ind.; F. Cajori, Colorado College; E. L. Brown, North High School, Denver, Colo.; H. E. Hawkes, Yale University. (3) Sub-committee on Exercises and Applications: Chairman, H. L. Reitz, The University of Illinois; Secretary, R. L. Short, Technical High School, Cleveland, Ohio.; Mabel Sykes, South Chicago High School, Chicago, Ill.; F. K. Newton, Andover, Mass.; H. E. Slaught, University of Chicago.

The work laid out for the three sub-committees is:

1. The considerations of axioms, definitions (including new terms, symbols, distribution, etc.), assumptions, informal proofs; treatment of limits and incommensurables; time and place in the curriculum; purpose; historical notes; and other related topics.

2. Consideration of courses in other countries; preliminary inductive courses; courses for different classes of students; minimum list of basal theorems for all students; full list of

theorems for college examinations; classification of theorems, and any other related topics.

3. Grading and distribution of exercises; relative importance of algebraic and geometrical exercises; special classes of exercises (as loci); correlation with other subjects, as trigonometry; concrete applications related to drawing, architecture, machinery, etc.; and any other related topics.

The committee has undertaken a much needed work, and as it will consider as a basis for its own work, all reports of sectional committees in this country, as well as what has been accomplished in foreign countries, this summing up of the best thought of the present day on the subject of geometry should carry weight in improving the teaching of the subject, as well as acting as a powerful unifying agent.

NEW BOOKS.

Plane and Spherical Trigonometry and Tables. By W. A. GRANVILLE. Boston: Ginn and Company. pp. 264 + 38. $1.25.

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A FEW ALGEBRA METHODS.

By Philip R. Dean.

Granted certain favorable conditions such as: (1) a thoroughly trained and broad-minded teacher, (2) reasonable-sized classes, and (3) pupils mature enough for the work at hand, two things seem to me highly essential for the best teaching. These are (*a*) well-grounded methods; and (*b*) daily and sympathetic planning of each lesson.

Good teaching is hard work even to the man who loves it intensely. We are before a class for forty-five minutes daily: are we realizing our opportunity if we lose time for them through our own poor administration of the work, or our failure to perceive at once any difficulty that arises and its remedy —or again, if we lack that "entire command" of the subject and fresh familiarity with the day's topics which insure the respect of the whole class?

It is said among public speakers that it is a long lesson to learn to extemporize well, and that the best preparation is to begin by writing out carefully the first speeches and learning them verbatim; after that fewer and fewer words need be written out ahead, but always the great extempore speech consists of material well thought out beforehand but fused afresh into glowing substance by the inspirational fire of the occasion.

So, too, must it be with musical improvisation. Lowell's "musing organist" "lets his fingers wander as they list" only

when he has mastered the technique and can depend on these mechanical means to express *sua sponte* his deeper feelings.

Is it not thus in our teaching? We must struggle, by long studied practice, to be able to reproduce, the first year perhaps, some one teaching method, which we have thought out carefully; then similarly other methods, until gradually we may move with confidence, and may come before a class with only the immediate subject matter vividly in mind.

This amounts to training the intuition coupled with a mastery of several recognized methods. An experienced resourceful teacher feels quickly when he is not succeeding. Happy he if, in his preparation, he has anticipated this particular situation and is ready with exactly *the* method needed.

I maintain that, even with large classes, it is well worth while to give several minutes before any recitation devising a method especially to reach some backward pupil without sacrificing the others. There is satisfaction in success of this kind; and indeed I have no sympathy with the teacher who says that such and such a pupil "cannot learn mathematics." That teacher is but admitting his own impotence.

George Herbert Palmer gives voice in one of his recent books, to the idea that the great teacher can only be he who delights in *teaching*, and who loves the work for its own sake. We cannot all be great teachers, but we can catch glimpses of the ideal if we can rise to sympathetic touch with the movements of our time and keep our own minds so keenly alive to the learner's view-point that we must ever try to give him our best,—even as *we* should expect from one who stood before *us* to instruct.

I have been in classes under college professors where I was morally certain the instructor had not made a minute's immediate preparation. I got nothing out of it and despised the man. Nay more, I have stood before classes myself when I did not really know, except as I stumbled on them accidentally, what points were the difficult ones to the pupils, nor how to go at these points. I could give the majority of the class perhaps nothing, and, as maybe beads came out on my forehead, I have despised myself in deep humility.

I believe that, with all rigor and exactness, mathematics should be taught psychologically rather than logically. Somewhere, in Preyer or Sully, I read years ago before I was

married, that the young child was certain "vastly to be influenced by his surroundings and in turn vastly to influence them." In my greenness I smiled at this last phrase for I could not see how. And yet as I think of it now, I believe the truest word ever said to teachers is—" And a little child shall lead them."

Methods, then, should not be looked on as a means of saving the teacher time and energy—but rather, each new one should be judged by its possibilities for increasing the teacher's efficiency.

I shall not attempt a summary of well-established standard methods such as the "chalk and talk" of Professor Young, and the methods of awakening interest and checking results discussed by Professor D. E. Smith. I can only mention a few not written so much about.

A good friend of mine, once in an important oral examination, was asked when the "concert method" should be used. He thought a minute and answered, "When the exercise is a written one!" Now this expressed exactly what we were trying, under Mr. Anthony, of DeWitt Clinton High School, six years ago, to do to get at every pupil in our crowded classes. The method was used at various stages of the work. As many pupils as possible were sent to the board; the others at their seats with paper. Dictation (all listening): "The sum of the cubes of a and b": a second's pause, a signal and all turn and write it quickly. Again: "All write x equals tens digit, y equals units digit:—Attention! Express in algebra the number." At signal all write it. "Attention! Express the number formed by doubling tens digit." Signal after pause and all write. "Attention! Write the algebraic statement that the number formed by doubling tens digit is 200 less than the square of the given number." Pause and then all write. If the questions are "thought-provoking" and the pauses properly timed any one in the room can see at a glance which pupils are doing the thinking and obtaining the answer unaided. Moreover, the teacher has ever before him evidence showing exactly how well the class, as a whole, is keeping up with the development of the topic, as he outlines it, and where drill is needed. We found this method especially good at introducing new branches of the work, and in drilling on algebraic symbolism or shorthand. It demands of the teacher, in a high degree, skill and keenness. He is put in a strategic position where